

# The boundedness of some operators with rough kernel on the weighted Morrey spaces

Hua Wang<sup>\*</sup>

School of Mathematical Sciences, Peking University, Beijing 100871, China

## Abstract

Let  $\Omega \in L^q(S^{n-1})$  with  $1 < q \leq \infty$  be homogeneous of degree zero and has mean value zero on  $S^{n-1}$ . In this paper, we will study the boundedness of homogeneous singular integrals and Marcinkiewicz integrals with rough kernel on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for  $q' \leq p < \infty$  (or  $q' < p < \infty$ ) and  $0 < \kappa < 1$ . We will also prove that the commutator operators formed by a  $BMO(\mathbb{R}^n)$  function  $b(x)$  and these rough operators are bounded on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for  $q' < p < \infty$  and  $0 < \kappa < 1$ .

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## 1. Introduction

Suppose that  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$  ( $n \geq 2$ ) equipped with the normalized Lebesgue measure  $d\sigma$ . Let  $\Omega \in L^q(S^{n-1})$  with  $1 < q \leq \infty$  be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where  $x' = x/|x|$  for any  $x \neq 0$ . The homogeneous singular integral operator  $T_\Omega$  is defined by

$$T_\Omega f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x - y) dy$$

and a related maximal operator  $M_\Omega$  is defined by

$$M_\Omega f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|y| < r} |\Omega(y') f(x - y)| dy.$$

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<sup>\*</sup>E-mail address: wanghua@pku.edu.cn.

Let  $b$  be a locally integrable function on  $\mathbb{R}^n$ , the commutator of  $b$  and  $T_\Omega$  is defined as follows

$$[b, T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x).$$

The Marcinkiewicz integral of higher dimension  $\mu_\Omega$  is defined by

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

It is well known that the Littlewood-Paley  $g$ -function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley  $g$ -function. In this paper, we will also consider the commutator  $[b, \mu_\Omega]$  which is given by the following expression

$$[b, \mu_\Omega]f(x) = \left( \int_0^\infty |F_{\Omega,t}^b(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}^b(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy.$$

The classical Morrey spaces  $\mathcal{L}^{p,\lambda}$  were first introduced by Morrey in [10] to study the local behavior of solutions to second order elliptic partial differential equations. Recently, Komori and Shirai [9] considered the weighted version of Morrey spaces  $L^{p,\kappa}(w)$  and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces.

The main purpose of this paper is to discuss the weighted boundedness of the above operators  $M_\Omega$ ,  $T_\Omega$  and  $\mu_\Omega$  with rough kernels on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for  $q' \leq p < \infty$  and  $0 < \kappa < 1$ , where we set the notation  $q' = q/(q-1)$  when  $1 < q < \infty$  and  $q' = 1$  when  $q = \infty$ . We shall also show that the commutators  $[b, T_\Omega]$  and  $[b, \mu_\Omega]$  are bounded operators on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for  $q' < p < \infty$  and  $0 < \kappa < 1$ , where the symbol  $b$  belongs to  $BMO(\mathbb{R}^n)$ . Our main results are stated as follows.

**Theorem 1.** *Assume that  $\Omega \in L^q(S^{n-1})$  with  $1 < q < \infty$ . Then for every  $q' \leq p < \infty$ ,  $w \in A_{p/q'}$  and  $0 < \kappa < 1$ , there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|M_\Omega(f)\|_{L^{p,\kappa}(w)} \leq C \|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 2.** Assume that  $\Omega \in L^q(S^{n-1})$  with  $1 < q < \infty$ . Then for every  $q' \leq p < \infty$ ,  $w \in A_{p/q'}$  and  $0 < \kappa < 1$ , there exists a constant  $C > 0$  independent of  $f$  such that

$$\|T_\Omega(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 3.** Assume that  $\Omega \in L^q(S^{n-1})$  with  $1 < q < \infty$  and  $b \in BMO(\mathbb{R}^n)$ . Then for every  $q' < p < \infty$ ,  $w \in A_{p/q'}$  and  $0 < \kappa < 1$ , there exists a constant  $C > 0$  independent of  $f$  such that

$$\|[b, T_\Omega](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 4.** Assume that  $\Omega \in L^q(S^{n-1})$  with  $1 < q \leq \infty$ . Then for every  $q' < p < \infty$ ,  $w \in A_{p/q'}$  and  $0 < \kappa < 1$ , there exists a constant  $C > 0$  independent of  $f$  such that

$$\|\mu_\Omega(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 5.** Assume that  $\Omega \in L^q(S^{n-1})$  with  $1 < q \leq \infty$  and  $b \in BMO(\mathbb{R}^n)$ . Then for every  $q' < p < \infty$ ,  $w \in A_{p/q'}$  and  $0 < \kappa < 1$ , there exists a constant  $C > 0$  independent of  $f$  such that

$$\|[b, \mu_\Omega](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

## 2. Notations and definitions

First let us recall some standard definitions and notations. The classical  $A_p$  weight theory was first introduced by Muckenhoupt in the study of weighted  $L^p$  boundedness of Hardy-Littlewood maximal functions in [11]. A weight  $w$  is a locally integrable function on  $\mathbb{R}^n$  which takes values in  $(0, \infty)$  almost everywhere,  $B = B(x_0, r)$  denotes the ball with the center  $x_0$  and radius  $r$ . We say that  $w \in A_p$ ,  $1 < p < \infty$ , if

$$\left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

where  $C$  is a positive constant which is independent of  $B$ .

For the case  $p = 1$ ,  $w \in A_1$ , if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess\,inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

A weight function  $w$  is said to belong to the reverse Hölder class  $RH_r$  if there exist two constants  $r > 1$  and  $C > 0$  such that the following reverse Hölder inequality holds

$$\left( \frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) dx \right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

It is well known that if  $w \in A_p$  with  $1 < p < \infty$ , then  $w \in A_r$  for all  $r > p$ , and  $w \in A_q$  for some  $1 < q < p$ . If  $w \in A_p$  with  $1 \leq p < \infty$ , then there exists  $r > 1$  such that  $w \in RH_r$ .

We give the following results that we will use frequently in the sequel.

**Lemma A** ([6]). *Let  $w \in A_p$ ,  $p \geq 1$ . Then, for any ball  $B$ , there exists an absolute constant  $C$  such that*

$$w(2B) \leq Cw(B).$$

*In general, for any  $\lambda > 1$ , we have*

$$w(\lambda B) \leq C\lambda^{np}w(B),$$

*where  $C$  does not depend on  $B$  nor on  $\lambda$ .*

**Lemma B** ([7]). *Let  $w \in RH_r$  with  $r > 1$ . Then there exists a constant  $C$  such that*

$$\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

*for any measurable subset  $E$  of a ball  $B$ .*

A locally integrable function  $b$  is said to be in  $BMO(\mathbb{R}^n)$  if

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty,$$

where  $b_B = \frac{1}{|B|} \int_B b(y) dy$  and the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ .

**Theorem C** ([5,8]). *Assume that  $b \in BMO(\mathbb{R}^n)$ . Then for any  $1 \leq p < \infty$ , we have*

$$\sup_B \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{1/p} \leq C\|b\|_*.$$

Next we shall define the weighted Morrey space and give one of the results relevant to this paper. For further details, we refer the readers to [9].

**Definition 1.** Let  $1 \leq p < \infty$ ,  $0 < \kappa < 1$  and  $w$  be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \{f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left( \frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{1/p}$$

and the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ .

In [9], the authors obtained the following result.

**Theorem D.** If  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ , then the Hardy-Littlewood maximal operator  $M$  is bounded on  $L^{p,\kappa}(w)$ .

We are going to conclude this section by giving several results concerning the weighted boundedness of rough operators  $M_\Omega$ ,  $T_\Omega$  and  $\mu_\Omega$  on the weighted  $L^p$  spaces. Given a Muckenhoupt's weight function  $w$  on  $\mathbb{R}^n$ , for  $1 \leq p < \infty$ , we denote by  $L^p_w(\mathbb{R}^n)$  the space of all functions satisfying

$$\|f\|_{L^p_w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

**Theorem E** ([4]). Suppose that  $\Omega \in L^q(S^{n-1})$ ,  $1 < q < \infty$ . Then for every  $q' \leq p < \infty$  and  $w \in A_{p/q'}$ , there is a constant  $C$  independent of  $f$  such that

$$\|M_\Omega(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}$$

$$\|T_\Omega(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.$$

**Theorem F** ([2]). Suppose that  $\Omega \in L^q(S^{n-1})$ ,  $1 < q \leq \infty$ . Then for every  $q' < p < \infty$  and  $w \in A_{p/q'}$ , there is a constant  $C$  independent of  $f$  such that

$$\|\mu_\Omega(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.$$

**Theorem G** ([3]). Suppose that  $\Omega \in L^q(S^{n-1})$  with  $1 < q \leq \infty$  and  $b \in BMO(\mathbb{R}^n)$ . Then for  $q' < p < \infty$  and  $w \in A_{p/q'}$ , there is a constant  $C > 0$  independent of  $f$  such that

$$\|[b, \mu_\Omega](f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.$$

Throughout this article, we will use  $C$  to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By  $A \sim B$ , we mean that there exists a constant  $C > 1$  such that  $\frac{1}{C} \leq \frac{A}{B} \leq C$ .

### 3. Proof of Theorem 1

First, by using Hölder's inequality, we can easily see that

$$M_\Omega f(x) \leq C \cdot \|\Omega\|_{L^q(S^{n-1})} M_{q'}(f)(x),$$

where  $M_{q'}(f)(x) = M(|f|^{q'})(x)^{1/q'}$ . Then for  $q' < p < \infty$  and  $w \in A_{p/q'}$ , it follows immediately from Theorem D that

$$\begin{aligned} \|M_{q'}(f)\|_{L^{p,\kappa}(w)} &= \|M(|f|^{q'})\|_{L^{p/q',\kappa}(w)}^{1/q'} \\ &\leq C \| |f|^{q'} \|_{L^{p/q',\kappa}(w)}^{1/q'} \\ &\leq C \|f\|_{L^{p,\kappa}(w)}. \end{aligned}$$

Now we consider the case  $p = q'$ . Fix a ball  $B = B(x_0, r_B) \subseteq \mathbb{R}^n$  and decompose  $f = f_1 + f_2$ , where  $f_1 = f\chi_{2B}$ ,  $\chi_{2B}$  denotes the characteristic function of  $2B$ . Since  $M_\Omega$  is a sublinear operator, then we have

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f_1(x)|^p w(x) dx \right)^{1/p} \\ &\quad + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f_2(x)|^p w(x) dx \right)^{1/p} \\ &= I_1 + I_2. \end{aligned}$$

Theorem E and Lemma A imply

$$\begin{aligned} I_1 &\leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\leq C \|f\|_{L^{p,\kappa}(w)}. \end{aligned} \tag{1}$$

We turn to estimate the term  $I_2$ . For any given  $r > 0$  and  $x \in B$ , by Hölder's inequality and the  $A_1$  condition, we thus obtain

$$\begin{aligned} &\frac{1}{r^n} \int_{|y|<r} |\Omega(y') f_2(x-y)| dy \\ &\leq \frac{1}{r^n} \left( \int_{|y|<r} |\Omega(y')|^q dy \right)^{1/q} \left( \int_{|y|<r} |f_2(x-y)|^p dy \right)^{1/p} \\ &\leq C \cdot \|\Omega\|_{L^q(S^{n-1})} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_2(y)|^p dy \right)^{1/p} \end{aligned}$$

$$\leq C \cdot \|\Omega\|_{L^q(S^{n-1})} \left( \frac{1}{w(B(x, r))} \int_{B(x, r)} |f_2(y)|^p w(y) dy \right)^{1/p}.$$

A simple geometric observation shows that when  $x \in B(x_0, r_B)$  and  $y \in B(x, r) \cap (2B(x_0, r_B))^c$ , then we have  $B(x_0, r_B) \subseteq 3B(x, r)$ . Hence

$$\begin{aligned} \frac{1}{r^n} \int_{|y| < r} |\Omega(y') f_2(x - y)| dy &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot \frac{1}{w(B(x, r))^{(1-\kappa)/p}} \\ &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot \frac{1}{w(B(x_0, r_B))^{(1-\kappa)/p}}. \end{aligned}$$

Taking the supremum over all  $r > 0$ , we can get

$$|M_\Omega(f_2)(x)| \leq C \|f\|_{L^{p, \kappa}(w)} \cdot \frac{1}{w(B(x_0, r_B))^{(1-\kappa)/p}},$$

which implies

$$I_2 \leq C \|f\|_{L^{p, \kappa}(w)}. \quad (2)$$

Combining the above inequality (2) with (1) and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we obtain the desired result.

#### 4. Proofs of Theorems 2 and 3

*Proof of Theorem 2.* Fix a ball  $B = B(x_0, r_B)$  and decompose  $f = f_1 + f_2$ , where  $f_1 = f\chi_{2B}$ . Then we have

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f_1(x)|^p w(x) dx \right)^{1/p} \\ &\quad + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f_2(x)|^p w(x) dx \right)^{1/p} \\ &= J_1 + J_2. \end{aligned}$$

Theorem E and Lemma A give

$$\begin{aligned} J_1 &\leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\leq C \|f\|_{L^{p, \kappa}(w)}. \end{aligned}$$

In order to estimate  $J_2$ , we first deduce from Hölder's inequality that

$$\begin{aligned} & |T_\Omega(f_2)(x)| \\ &= \left| \int_{(2B)^c} \frac{\Omega((x-y)')}{|x-y|^n} f(y) dy \right| \\ &\leq \sum_{j=1}^{\infty} \left( \int_{2^{j+1}B \setminus 2^jB} |\Omega((x-y)')|^q dy \right)^{1/q} \left( \int_{2^{j+1}B \setminus 2^jB} \frac{|f(y)|^{q'}}{|x-y|^{nq'}} dy \right)^{1/q'}. \end{aligned}$$

When  $x \in B$  and  $y \in 2^{j+1}B \setminus 2^jB$ , then by a direct calculation, we can see that  $2^{j-1}r_B \leq |y-x| < 2^{j+2}r_B$ . Hence

$$\left( \int_{2^{j+1}B \setminus 2^jB} |\Omega((x-y)')|^q dy \right)^{1/q} \leq C \cdot \|\Omega\|_{L^q(S^{n-1})} |2^{j+1}B|^{1/q}. \quad (3)$$

We also note that if  $x \in B$ ,  $y \in (2B)^c$ , then  $|y-x| \sim |y-x_0|$ . Consequently

$$\left( \int_{2^{j+1}B \setminus 2^jB} \frac{|f(y)|^{q'}}{|x-y|^{nq'}} dy \right)^{1/q'} \leq \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |f(y)|^{q'} dy \right)^{1/q'}.$$

So we have

$$|T_\Omega(f_2)(x)| \leq C \sum_{j=1}^{\infty} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} dy \right)^{1/q'}.$$

We shall consider two cases. When  $p = q'$ , then by the  $A_1$  condition, we get

$$\begin{aligned} |T_\Omega(f_2)(x)| &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}. \end{aligned} \quad (4)$$

When  $p > q'$ , set  $s = p/q' > 1$ . Then it follows from the Hölder's inequality and the  $A_s$  condition that

$$\begin{aligned} |T_\Omega(f_2)(x)| &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1/q'}} \left( \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\ &\quad \times \left( \int_{2^{j+1}B} w^{-s'/s}(y) dy \right)^{1/s'q'} \\ &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}. \end{aligned} \quad (5)$$



Hence, for every  $q' \leq p < \infty$ , by the estimates (4) and (5), we obtain

$$J_2 \leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \left( \frac{w(B)}{w(2^{j+1}B)} \right)^{(1-\kappa)/p}.$$

Since  $w \in A_{p/q'}$ , then there exists  $r > 1$  such that  $w \in RH_r$ . By using Lemma B, we thus get

$$\frac{w(B)}{w(2^{j+1}B)} \leq C \left( \frac{|B|}{|2^{j+1}B|} \right)^{(r-1)/r}. \quad (6)$$

Therefore

$$\begin{aligned} J_2 &\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \left( \frac{1}{2^{jn}} \right)^{(1-\kappa)(r-1)/pr} \\ &\leq C \|f\|_{L^{p,\kappa}(w)}, \end{aligned}$$

where the last series is convergent since  $(1-\kappa)(r-1)/pr > 0$ . Using the estimates for  $J_1$  and  $J_2$  and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we complete the proof of Theorem 2.  $\square$

*Proof of Theorem 3.* As in the proof of Theorem 2, we can write

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, T_{\Omega}]f(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, T_{\Omega}]f_1(x)|^p w(x) dx \right)^{1/p} \\ &\quad + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, T_{\Omega}]f_2(x)|^p w(x) dx \right)^{1/p} \\ &= J'_1 + J'_2. \end{aligned}$$

By Theorem E and the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Bagby, Kurtz and Pérez (see [1]), we see that  $[b, T_{\Omega}]$  is bounded on  $L_w^p$  for all  $q' < p < \infty$  and  $w \in A_{p/q'}$ . This together with Lemma A yield

$$\begin{aligned} J'_1 &\leq C \|b\|_* \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \end{aligned} \quad (7)$$

We now turn to deal with the term  $J'_2$ . For any given  $x \in B$ , we have

$$\begin{aligned} |[b, T_\Omega]f_2(x)| &\leq |b(x) - b_B| \cdot \int_{(2B)^c} \frac{|\Omega((x-y)')|}{|x-y|^n} |f(y)| dy \\ &\quad + \int_{(2B)^c} \frac{|\Omega((x-y)')|}{|x-y|^n} |b(y) - b_B| |f(y)| dy \\ &= \text{I} + \text{II}. \end{aligned}$$

In the proof of Theorem 2, for any  $q' < p < \infty$ , we have already showed

$$\text{I} \leq C|b(x) - b_B| \cdot \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}.$$

Consequently

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \frac{1}{w(B)^{\kappa/p}} \cdot \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}} \cdot \left( \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \\ &= C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \cdot \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p}. \end{aligned}$$

Using the same arguments as that of Theorem 2, we can see that the above summation is bounded by a constant. Hence

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \leq C \|f\|_{L^{p,\kappa}(w)} \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p}.$$

Since  $w \in A_{p/q'}$ , then  $w \in A_p$ . As before, there exists a number  $r > 1$  such that  $w \in RH_r$ . By the reverse Hölder's inequality and Theorem C, we get

$$\begin{aligned} &\left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \\ &\leq C \cdot \frac{1}{w(B)^{1/p}} \left( \int_B |b(x) - b_B|^{pr'} dx \right)^{1/pr'} \left( \int_B w(x)^r dx \right)^{1/pr} \\ &\leq C \cdot \left( \frac{1}{|B|} \int_B |b(x) - b_B|^{pr'} dx \right)^{1/pr'} \\ &\leq C \|b\|_*. \end{aligned} \tag{8}$$

So we have

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \mathbb{I}^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (9)$$

On the other hand, it follows from Hölder's inequality and (3) that

$$\begin{aligned} \text{II} &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_B|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \\ &\quad + C \sum_{j=1}^{\infty} |b_{2^{j+1}B} - b_B| \cdot \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} dy \right)^{1/q'} \\ &= \text{III} + \text{IV}. \end{aligned}$$

Set  $s = p/q' > 1$ . Then by using Hölder's inequality, we thus obtain

$$\begin{aligned} &\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \quad (10) \\ &\leq \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q's'} w^{-s'/s}(y) dy \right)^{1/q's'} \left( \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{j+1}B)^{\kappa/p} \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q's'} w^{-s'/s}(y) dy \right)^{1/q's'}. \end{aligned}$$

Let  $v(y) = w^{-s'/s}(y) = w^{1-s'}(y)$ . Then we have  $v \in A_{s'}$  because  $w \in A_s$  (see [6]), which implies  $v \in A_{q's'}$ . Following along the same lines as that of (8), we can get

$$\left( \frac{1}{v(2^{j+1}B)} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q's'} v(y) dy \right)^{1/q's'} \leq C \|b\|_*. \quad (11)$$

Substituting the above inequality (11) into (10), we thus have

$$\begin{aligned} &\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{j+1}B)^{\kappa/p} \cdot v(2^{j+1}B)^{1/q's'} \quad (12) \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot |2^{j+1}B|^{1/q'} \cdot w(2^{j+1}B)^{(\kappa-1)/p}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{w(B)^{\kappa/p}} \left( \int_B \mathbb{III}^p w(x) dx \right)^{1/p} &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (13) \end{aligned}$$

Now let's deal with the last term IV. Since  $b \in BMO(\mathbb{R}^n)$ , then a simple computation shows that

$$|b_{2^{j+1}B} - b_B| \leq C \cdot j \|b\|_*. \quad (14)$$

It follows immediately from the inequalities (5) and (14) that

$$\begin{aligned} \text{IV} &\leq C \|b\|_* \sum_{j=1}^{\infty} j \cdot \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \cdot w(2^{j+1}B)^{(\kappa-1)/p}. \end{aligned}$$

Therefore, by the estimate (6), we obtain

$$\begin{aligned} \frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{IV}^p w(x) dx \right)^{1/p} &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \cdot \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{j}{2^{jn\theta}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}, \end{aligned} \quad (15)$$

where  $w \in RH_r$  and  $\theta = (1 - \kappa)(r - 1)/pr$ . Summarizing the estimates (13) and (15) derived above, we can get

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{IV}^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (16)$$

Combining the inequalities (7), (9) with the inequality (16) and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we conclude the proof of Theorem 3.  $\square$

## 5. Proofs of Theorems 4 and 5

*Proof of Theorem 4.* Fix a ball  $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ . Let  $f = f_1 + f_2$ , where  $f_1 = f \chi_{2B}$ . Then we have

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mu_{\Omega} f(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mu_{\Omega} f_1(x)|^p w(x) dx \right)^{1/p} \\ &\quad + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mu_{\Omega} f_2(x)|^p w(x) dx \right)^{1/p} \\ &= K_1 + K_2. \end{aligned}$$

Theorem F and Lemma A imply

$$\begin{aligned}
K_1 &\leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\
&\leq C \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\
&\leq C \|f\|_{L^{p,\kappa}(w)}.
\end{aligned}$$

To estimate  $K_2$ , observe that when  $x \in B$  and  $y \in 2^{j+1}B \setminus 2^j B$  ( $j \geq 1$ ), then

$$t \geq |x - y| \geq |y - x_0| - |x - x_0| \geq 2^{j-1}r_B.$$

Therefore

$$\begin{aligned}
|\mu_\Omega(f_2)(x)| &= \left( \int_0^\infty \left| \int_{(2B)^c \cap \{y: |x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq \sum_{j=1}^\infty \left( \int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \right) \cdot \left( \int_{2^{j-1}r_B}^\infty \frac{dt}{t^3} \right)^{1/2} \\
&\leq C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy.
\end{aligned}$$

When  $\Omega \in L^\infty(S^{n-1})$ , then by assumption, we have  $w \in A_p$ ,  $1 < p < \infty$ . It follows from the Hölder's inequality and the  $A_p$  condition that

$$\begin{aligned}
|\mu_\Omega(f_2)(x)| &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \cdot \frac{1}{|2^{j+1}B|^{(n-1)/n}} \int_{2^{j+1}B} |f(y)| dy \\
&\leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\
&\quad \times \left( \int_{2^{j+1}B} w(y)^{-p'/p} dy \right)^{1/p'} \\
&\leq C \|\Omega\|_{L^\infty(S^{n-1})} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^\infty w(2^{j+1}B)^{(\kappa-1)/p}.
\end{aligned} \tag{17}$$

When  $\Omega \in L^q(S^{n-1})$ ,  $1 < q < \infty$ , by the inequalities (3) and (5), we get

$$\begin{aligned}
|\mu_\Omega(f_2)(x)| &\leq C \|\Omega\|_{L^q(S^{n-1})} \sum_{j=1}^\infty \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} dy \right)^{1/q'} \\
&\leq C \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^\infty w(2^{j+1}B)^{(\kappa-1)/p}.
\end{aligned} \tag{18}$$

Hence, for  $1 < q \leq \infty$ ,  $q' < p < \infty$ , by the estimates (17) and (18), we have

$$\begin{aligned} K_2 &\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \left( \frac{w(B)}{w(2^{j+1}B)} \right)^{(1-\kappa)/p} \\ &\leq C \|f\|_{L^{p,\kappa}(w)}. \end{aligned}$$

Using the above estimates for  $K_1$  and  $K_2$  and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we get our desired result.  $\square$

*Proof of Theorem 5.* As before, we can write

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu_\Omega]f(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu_\Omega]f_1(x)|^p w(x) dx \right)^{1/p} \\ &\quad + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu_\Omega]f_2(x)|^p w(x) dx \right)^{1/p} \\ &= K'_1 + K'_2. \end{aligned}$$

Theorem G and Lemma A yield

$$\begin{aligned} K'_1 &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \end{aligned}$$

Finally, let us deal with the term  $K'_2$ . For any fixed  $x \in B$ , we have

$$\begin{aligned} &|[b, \mu_\Omega]f_2(x)| \\ &\leq |b(x) - b_B| \left( \int_0^\infty \left| \int_{(2B)^c \cap \{y: |x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad + \left( \int_0^\infty \left| \int_{(2B)^c \cap \{y: |x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(y) - b_B] f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &= \text{I} + \text{II}. \end{aligned}$$

In the proof of Theorem 4, for any  $q' < p < \infty$ , we have already proved

$$\text{I} \leq C |b(x) - b_B| \cdot \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} w(2^{j+1}B)^{(\kappa-1)/p}.$$

Following the same lines as in the proof of Theorem 3, we obtain

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \mathbb{I}^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}.$$

On the other hand, we note that when  $x \in B$  and  $y \in 2^{j+1}B \setminus 2^jB$  ( $j \geq 1$ ), then we have  $t \geq 2^{j-1}r_B$ . Consequently

$$\begin{aligned} \text{II} &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(y) - b_B| |f(y)| dy \\ &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(y) - b_{2^{j+1}B}| |f(y)| dy \\ &\quad + C \sum_{j=1}^{\infty} \frac{|b_{2^{j+1}B} - b_B|}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \\ &= \text{III} + \text{IV}. \end{aligned}$$

When  $\Omega \in L^\infty(S^{n-1})$ , then it follows from Hölder's inequality that

$$\begin{aligned} \text{III} &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| |f(y)| dy \\ &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} w^{-p'/p}(y) dy \right)^{1/p'} \\ &\quad \times \left( \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{1/p} \\ &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \cdot w(2^{j+1}B)^{\kappa/p} \\ &\quad \times \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} w^{-p'/p}(y) dy \right)^{1/p'}. \end{aligned}$$

Set  $u(y) = w^{-p'/p}(y) = w^{1-p'}(y)$ . In this case, since  $w \in A_p$ , then we have  $u \in A_{p'}$ , it follows from the inequality (8) and the  $A_p$  condition that

$$\begin{aligned} \text{III} &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} w(2^{j+1}B)^{\kappa/p} \cdot u(2^{j+1}B)^{1/p'} \\ &\leq C \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot \sum_{j=1}^{\infty} w(2^{j+1}B)^{(\kappa-1)/p}. \end{aligned} \tag{19}$$

When  $\Omega \in L^q(S^{n-1})$ , then by using Hölder's inequality, the inequalities (3) and (12), we can deduce

$$\begin{aligned} \text{III} &\leq C \|\Omega\|_{L^q(S^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1/q'}} \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C \|\Omega\|_{L^q(S^{n-1})} \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot \sum_{j=1}^{\infty} w(2^{j+1}B)^{(\kappa-1)/p}. \end{aligned} \quad (20)$$

Hence, for  $1 < q \leq \infty$ ,  $q' < p < \infty$ , by the estimates (19) and (20), we get

$$\begin{aligned} \frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{III}^p w(x) dx \right)^{1/p} &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \end{aligned}$$

Again, in Theorem 4, we have already obtained the following inequality

$$\frac{1}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \leq C \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{j+1}B)^{(\kappa-1)/p}.$$

From (14), it follows immediately that

$$\text{IV} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \cdot w(2^{j+1}B)^{(\kappa-1)/p}.$$

The rest of the proof is exactly the same as that of (15), we finally obtain

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{IV}^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}.$$

Therefore, by combining the above estimates and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we conclude the proof of Theorem 5.  $\square$

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